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A WRAPPED EXPONENTIAL CIRCULAR MODEL

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In this paper we have introduced and studied a new class of circular distributions resulting from wrapping the exponential distributions on the real line. The densities and distribution functions of wrapped exponential distributions admit explicit forms, as do trigonometric moments and related parameters. Wrapped exponential distributions retain the important properties of infinite divisibility and maxim entropy of the corresponding exponential distributions. The estimation of parameters via maximum likelihood is straightforward. The mixture of two wrapped exponential distributions leads to a wrapped Laplace distribution, so that the properties of former distribution are useful in studying the latter one. Both distributions are promising for modeling directional data.

Introduction

In this paper we obtain a new class of non-symmetric circular distributions by wrapping an exponential distribution around the circumference of a unit circle. When a real random variable (r. v.) X with probability density function (p.d.f.) f and characteristic function (ch. f.) ϕ is wrapped, then the p.d.f. of the wrapped r.v.,

$$X_w = X \ (\mathrm{mod}\ 2\pi) \tag{1}$$

has the density function

$$f_w(\theta) = \sum_{k=-\infty}^{\infty} f(\theta + 2k\pi), \theta \in [0, 2\pi). \tag{2}$$

and the characteristic function (the discrete Fourier transform)

$$\phi_p = Ee^{ipX_w} = \phi(p), p = 0, \pm 1, \pm 2, ... \tag{3}$$

(Jammalamadaka and Sen Gupta, 2000, Mardia and Jupp, 2000). Since the ϕ_p's are the Fourier coefficients, using what corresponds to the inversion formula, one can also write the density $f_w(\theta)$ in the alternative form

$$f_w(\theta) = \sum_p \phi_p e^{-ip\theta}, \theta \in [0, 2\pi). \tag{4}$$

Circular distributions play an important role in modeling directional data arising in various fields. While the wrapped normal, Cauchy, and stable distributions have been studied extensively (Levy,1939, Gatto and Jammalamadaka, 1999) little work seems to have been done for the case of wrapped exponential or double exponential (Laplace) distributions.

The exponential distribution, with its important memoryless property, is a standard model in reliability, queuing theory, and numerous other fields (Barlow and

Proschen , 1996), while the Laplace distribution and its generalizations, that arise in geometric summation, are becoming promising, particularly for financial applications (Kotz et al, 2000). We believe that the circular counterparts of the two families would find applications for directional data. In this paper we develop a basic theory for the wrapped exponential distribution, finding that many properties of exponential distribution on the real line have analogs in the directional setting. The wrapped Laplace distribution, which is a mixture of two wrapped exponential distributions, will be the subject of a forth coming sequel.

Wrapped exponential distribution

Consider an exponential distribution with parameter $\lambda > 0$, whose p.d.f. and ch.f. are

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0, \tag{5}$$

and

$$\phi(t) = 1/(1 - it/\lambda), \quad t \in R, \tag{6}$$

respectively. Then, by (3), the ch.f. of the corresponding wrapped distribution is

$$\phi_p = 1/(1 - ip/\lambda), \quad p = 0, \pm 1, \pm 2, \ldots.. \tag{7}$$

To derive the p.d.f corresponding to (7), use (2) and note that for $\theta \in [0, 2\pi)$ the density $f(\theta + 2k\pi)$ is non-zero only for $k \geq 0$. Thus, we obtain the following wrapped exponential p.d.f.

$$f_w(\theta) = \lambda e^{-\lambda\theta} \sum_{k=0}^{\infty} \left[e^{-2\pi\lambda}\right]^k = \frac{\lambda e^{-\lambda\theta}}{1 - e^{-2\pi\lambda}}, \quad \theta \in [0, 2\pi). \tag{8}$$

The above formula may be extended through the property of periodicity for values of θ outside the interval $[0, 2\pi)$, i.e.,

$$f_w(\theta) = f_w(\theta \bmod 2\pi) \tag{9}$$

for any other θ. That the density satisfies this periodicity property (9) can be checked easily.

Definition 1: *A r. v. Θ on the unit circle is said to have wrapped exponential distribution with parameter $\lambda > 0$, denoted by W E (λ), if the ch.f. and the p.d.f. of Θ are given by (7) and (8), respectively. We then write $\Theta \sim$ W E (λ).*

The distribution function (c.d.f.) of the wrapped exponential distribution is obtained easily by integrating the p.d.f.(8):

$$F_w(\theta) = \frac{1 - e^{-\lambda\theta}}{1 - e^{-2\pi\lambda}}, \quad \theta \in [0, 2\pi). \tag{10}$$

Observe that when $\lambda \to 0$ the above c.d.f. converges to $\theta/2\pi$, which is the c.d.f. of the circular uniform distribution. Thus, we can include the latter distribution along with the parameter $\lambda = 0$ into the above class.

Remark : Note that if the original exponential r.v. is restricted to the interval $[0, 2\pi)$, then the resulting distribution coincides with the wrapped distribution! In other words, the wrapped r.v. X_e given by (1) has the same probability distribution as $(X/X \leq 2\pi)$ for an exponentially distributed X (the truncated exponential distribution).

Remark : Notice that the wrapped exponential p.d.f.(8) integrates to 1 on $[0,2\pi)$ for any $\lambda \in R$. When $\lambda < 0$, the distribution given by (8) results from wrapping the *negative* exponential distribution with parameter $|\lambda|>0$, whose p.d.f. is

$$f(x)=|\lambda|e^{|\lambda|x},\ x<0. \tag{11}$$

Thus, one can consider a more general class of distributions $WE(\lambda)$ with $\lambda \in R$.

Remark: A more general class of circular distriburions can be obtained by wrapping the gamma distribution with p.d.f.

$$f(x)=\frac{\lambda^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}, x>0, \lambda>0, \alpha>0 \tag{12}$$

and characteristic function

$$\phi(t)=\left(\frac{1}{1-it/\lambda}\right)^{\alpha},\quad t\in R. \tag{13}$$

For $\alpha = 1$ the p.d.f (12) reduces to the exponential p.d.f. (5). Although the characteristic function of the corresponding wrapped distribution has a simple form,

$$\phi_p=\left(\frac{1}{1-ip/\lambda}\right)^{\alpha};\quad p=0,\pm1,\pm2,\ldots, \tag{14}$$

the corresponding density function,

$$f_w(\theta)=\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\sum_{k=0}^{\infty}(\theta+2k\pi)^{\alpha-1}e^{-2\pi\lambda k},\quad \theta\in[0,2\pi), \tag{15}$$

obtained by (2), lacks an explicit form (except for few special cases, including exponential with $\alpha = 1$).

Remark: Recall that for a given probability distribution density $p(x)$, the so called tilted (or weighted) distribution corresponding to the weight function $w(x)$ has density proportional to $w(x)\,p(x)$ (see, Rao, 1983). It was noted by Roy (1997) that the distribution with density (8) arises in logistic discrimination analysis as an exponentially weighted uniform circular distribution (with the weight function $w(x) = e^{-\lambda x}$).

Remark: By Wrapping a skewed Laplace distribution with density

$$f(x) = pf_1(x) + (1-p)f_2(x), \qquad x \in R, \tag{16}$$

where

$$f_1(x) = \lambda_1 e^{-\lambda_1 x} \ (x > 0)\ , \ f_2(x) = \lambda_2 e^{\lambda_2 x} \ (x < 0)\ , \tag{17}$$

and $p = 1/(k^2+1)\ ,\ \lambda_1 = \lambda k,\ \lambda_2 = \lambda / k$ (18)

for some $\kappa, \lambda > 0$ [Kotz et al, 2000) we obtain a skewed wrapped Laplace distribution with the explicit density

$$f_w(\theta) = p\frac{\lambda_1 e^{-\lambda_1 \theta}}{1-e^{-2\pi\lambda_1}} + (1-p)\frac{\lambda_2 e^{\lambda_2 \theta}}{e^{2\pi\lambda_2} - 1}, \theta \in [0, 2\pi). \tag{19}$$

Properties of the wrapped Laplace distribution with density (19) may be derived from those of wrapped exponential distribution.

Let $\lambda > 0$ and let Θ have a $WE(\lambda)$ distribution with density f given by (8). Then, the density of the r.v. $2\pi - \Theta$ is given by $f(2\pi - \theta)$. $\theta, \in [0, 2\pi]$. But,

$$f(2\pi - \theta) = \frac{\lambda e^{-\lambda(2\pi-\theta)}}{1-e^{-2\pi\lambda}} = \frac{-\lambda e^{\lambda\theta}}{1-e^{2\pi\lambda}}. \tag{20}$$

which is the density corresponding to the $WE(-\lambda)$ distribution. Thus we have the following simple result.

Lemma 1: *If* $\Theta \sim WE(\lambda)$ *then* $2\pi - \Theta \sim WE(-\lambda)$.

Note that if $\lambda > 0$ (wrapped exponential distributions), then the densities are strictly decreasing on the interval $[0, 2\pi)$, while for $\lambda < 0$ (wrapped negative exponential distributions) the densities are strictly increasing on $[0, 2\pi)$. All densities are defined on the whole real line by a periodic extension.

Trigonometric Moments and Related Parameters

We shall start with raw trigonometric moments α_p and β_p, defined by the equality

$$\phi_p = \alpha_p + i\beta_p\ ,\ p = 0, \pm 1, \pm 2, \tag{21}$$

Since the ch.f. of the $WE(\lambda)$ distribution with $\lambda \neq 0$ is

$$\phi_p = \frac{1}{1 - ip/\lambda} = \frac{\lambda^2}{\lambda^2 + p^2} + i\frac{p\lambda}{\lambda^2 + p^2}\ , \qquad p = 0, \pm 1, \pm 2, \tag{22}$$

we obtain

$$\alpha_p = \frac{\lambda^2}{\lambda^2 + p^2}, \ \beta_p = \frac{p\lambda}{\lambda^2 + p^2}, \qquad p = 0, \pm 1, \pm 2, \tag{23}$$

Remark: By the representation (4), the wrapped exponential density admits the Fourier representation

$$f_w(\theta) = \frac{1}{2\pi}\left[1 + 2\sum_{p=1}^{\infty} \frac{\lambda^2}{\lambda^2 + p^2} \cos p\theta + \frac{\lambda p}{\lambda^2 + p^2} \sin p\theta \right]. \tag{24}$$

The mean direction and the resultant length: Now, we write

$$\phi_p = \rho_p e^{i\mu_p^0}, \ p \geq 0, \tag{25}$$

so that

$$\rho_p = \sqrt{\alpha_p^2 + \beta_p^2} = |\lambda| / \sqrt{\lambda^2 + p^2}, \ p \geq 0. \tag{26}$$

The angle μ_p^0 satisfies the equation

$$\tan \mu_p^0 = \frac{\beta_p}{\alpha_p} = \frac{p}{\lambda}, \tag{27}$$

so that we have

$$\mu_p^0 = \begin{cases} \tan^{-1}(p/\lambda), & \text{for } \lambda > 0, \\ 2\pi + \tan^{-1}(p/\lambda), & \text{for } \lambda < 0. \end{cases} \tag{28}$$

In particular, the resultant length is

$$\rho = \rho_1 = \frac{|\lambda|}{\sqrt{1 + \lambda^2}}, \tag{29}$$

while the mean direction is

$$\mu_0 = \mu_1^0 = \begin{cases} \tan^{-1}(1/\lambda), & \text{for } \lambda > 0, \\ 2\pi + \tan^{-1}(1/\lambda), & \text{for } \lambda < 0. \end{cases} \tag{30}$$

Note that $\mu_0 \in (0, \pi/2)$ for $\lambda > 0$ and $\mu_0 \in (3\pi/2, 2\pi)$ for $\lambda < 0$.

Circular variance and standard deviation: The circular variance $V_0 = 1 - \rho$, becomes

$$V_0 = \frac{\sqrt{1+\lambda^2} - |\lambda|}{\sqrt{1+\lambda^2}} \tag{31}$$

The circular standard deviation,

$$\sigma_0 = \sqrt{-2\, ln(1 - V_0)}, \tag{32}$$

takes the form

$$\sigma_0 = \sqrt{ln\left(1 + 1/\lambda^2\right)} \tag{33}$$

Central trigonometric moments: The central trigonometric moments of a circular r.v. Θ are defined by the equation

$$\mu_p = E\, e^{ip(\Theta - \mu_0)} = \overline{\alpha_p} + i\, \overline{\beta_p}\ , \tag{34}$$

where μ_0 is the mean direction. Since the expectation in (34) is given by $\phi_p e^{-ip\mu_0}$, we have

$$\overline{\alpha_p} = \rho_p\, cos\left(\mu_p^0 - p\mu_0\right), \overline{\beta_p} = \rho_p\, sin\left(\mu_p^0 - p\mu_0\right). \tag{35}$$

By (26) and (28), the central trigonometric moments of the *WE* (λ) distribution with $\lambda \neq 0$ are

$$\overline{\alpha_p} = \frac{|\lambda|}{\sqrt{\lambda^2 + p^2}} \cos\left(\tan^{-1}(p/\lambda) - p\tan^{-1}(1/\lambda)\right) \tag{36}$$

and

$$\overline{\beta_p} = \frac{|\lambda|}{\sqrt{\lambda^2 + p^2}} sin\left(tan^{-1}(p/\lambda) - p\, tan^{-1}(1/\lambda)\right). \tag{37}$$

Skewness and Kurtosis : One common measure of skeweness of a circular distribution is

$$\gamma_1^0 = \overline{\beta_2}/V_0^{3/2}\ , \tag{38}$$

(Mardia and Jupp, 2000). To calculate γ_1^0 for the wrapped exponential distribution, we need its second central trigonometric moment. By (35), we have

$$\overline{\beta_2} = \frac{|\lambda|}{\sqrt{\lambda^2 + 4}}\left(sin\mu_2^0\, cos\, 2\mu_0 - cos\, \mu_2^0\, sin\, 2\mu_0\right). \tag{39}$$

Since

$$\sin\mu_2^0 = \frac{\beta_2}{\rho_2} = \frac{2\sin\lambda}{\sqrt{\lambda^2+4}}, \quad \cos\mu_2^0 = \frac{\alpha_2}{\rho_2} = \frac{|\lambda|}{\sqrt{4+\lambda^2}} \tag{40}$$

$$\sin 2\mu_0 = 2\sin\mu_0\cos\mu_0 = \frac{\beta_1}{\rho_1}\frac{\alpha_1}{\rho_1} = \frac{2\lambda}{1+\lambda^2} \text{ and } \cos 2\mu_0 = 1-2\sin^2\mu_0 = 1-\frac{2}{1+\lambda^2}, \tag{41}$$

we obtain after some algebra

$$\overline{\beta_2} = \frac{-2\lambda}{(1+\lambda^2)(4+\lambda^2)}. \tag{42}$$

Thus, the skewness of the *WE* (λ) distribution is

$$\gamma_1^0 = \frac{-2\lambda}{(1+\lambda^2)^{1/2}(4+\lambda^2)\left(\sqrt{1+\lambda^2}-\lambda\right)^{3/2}}. \tag{43}$$

Next, we consider the circular kurtosis,

$$\gamma_2^0 = \frac{\overline{\alpha_2}-(1-V_0)^4}{V_0^2}, \tag{44}$$

(Mardia and Jupp, 2000). The value of $\overline{\alpha}_2$ follows from (35),

$$\overline{\alpha_2} = \frac{|\lambda|}{\sqrt{\lambda^2+4}}\left(\cos\mu_2^0\cos 2\mu_0 + \sin\mu_2^0\sin 2\mu_0\right). \tag{45}$$

The substitution of (40) and (41) into (45) results in

$$\overline{\alpha_2} = \frac{\lambda^2(\lambda^2+3)}{(1+\lambda^2)(4+\lambda^2)}. \tag{46}$$

Now, we substitute (46) along with (31) into (44) to obtain after routine calculations :

$$\gamma_2^0 = \frac{3\lambda^2}{(1+\lambda^2)(4+\lambda^2)\left(\sqrt{1+\lambda^2}-|\lambda|\right)^2}. \tag{47}$$

We see that the distribution is leptokurtic ($\gamma_2^0 > 0$) for all $\lambda \neq 0$.

The median direction: A population median direction ξ_0 of a circular distribution with density *f* is any solution (in the interval $[0, 2\pi)$) of

$$\int_{\xi_0}^{\xi_0+\pi} f(\theta)\,d\theta = \int_{\xi_0+\pi}^{\xi_0+2\pi} f(\theta)\,d\theta = 1/2 \;, \qquad (48)$$

where the density f satisfies

$$f(\xi_0) > f(\xi_0+\pi) \;. \qquad (49)$$

(Mardia and Jupp, 2000). The calculation of the median direction for the wrapped exponential distribution is straight forward. For $\xi \in [0,\pi]$ define a function

$$g(\xi) = \int_{\xi}^{\xi+\pi} f_w(\theta)\,d\theta \;. \qquad (50)$$

where f_w is the density (8) of the $WE(\lambda)$ distribution. Note that for $\lambda > 0$ the function g is continuous and strictly decreasing with $g(0) > 1/2$ and $g(\pi) < 1/2$, while for $\lambda < 0$, it is continuous and strictly increasing with $g(0) < 1/2$ and $g(\pi) > 1/2$. Thus, the equation (48) has a unique solution. Substituting the $WE(\lambda)$ density (8) into (48) and taking into account (49), after routine calculations we obtain the following values of the median direction:

$$\xi_0 = \frac{1}{\lambda} \ln \frac{2}{1+e^{-\lambda\pi}} + \begin{cases} 0, & \text{for } \lambda > 0, \\ \pi, & \text{for } \lambda < 0. \end{cases} \qquad (51)$$

Properties

In this section we shall study further properties of the wrapped exponential distribution.

Infinite divisibility: Recall that an angular r.v. Θ (and its probability distribution) is said to be infinitely divisible if for any integer $n \geq 1$ there exist i.i.d. angular r.v.'s $\Theta_1, \ldots, \Theta_n$ such that

$$\Theta_1 + \ldots\ldots + \Theta_n \;(\mathrm{mod}\; 2\pi) \stackrel{d}{=} \Theta \;. \qquad (52)$$

As remarked by Mardia and Jupp (2000), if a real r.v. X is infinitely divisible, then so is the wrapped r.v. X_w. Thus, since an exponential r.v. X is infinitely divisible (as is $-X$), the wrapped exponential distribution $WE(\lambda)$ is infinitely divisible for every $\lambda \in R$. Indeed, if $\lambda = 0$, then the $WE(0)$ r.v. Θ has the circular uniform distributions and thus satisfies (52), where Θ's are i.i.d copies of Θ (the convolution of circular uniform distributions is circular uniform). Alternatively, if $\Theta \sim WE(\lambda)$ with $\lambda \neq 0$, then the ch.f. of Θ, ϕ_p, can be factored as

$$\phi_p = \frac{1}{1 - ip/\lambda} = (\tilde{\phi}_p)^n \;, \qquad (53)$$

where

$$\tilde{\phi}_p = \left(\frac{1}{1 - ip/\lambda} \right)^{1/n} \tag{54}$$

is the ch.f. of the wrapped gamma distribution. Since the factorization (53) of the ch.f. of Θ is equivalent to (52), we see that the wrapped exponential distribution is infinitely divisible. The following result summarizes these discussion.

Proposition 1: *If Θ ~ WE (λ), where λ ∈ R, then Θ is infinitely divisible. Moreover, for any positive integer n ≥ 1, the equality in distribution (52) holds where the* Θ_i*'s have the uniform circular distribution for λ = 0 and the wrapped gamma distribution with the ch.f (54) for λ ≠ 0.*

Geometric infinite divisibility : The classical exponential distribution has an important property of stability with respect to geometric compounding. Let $\{v_q, q \in (0,1)\}$be a family of geometric distributions with mean $1/q$, so that

$$P(v_q = k) = (1-q)^{k-1} q, \quad k = 1, 2, 3, \ldots\ldots \tag{55}$$

As noted by Arnold (1973), if X has an exponential distribution with parameter λ and density (5) then.

$$q \sum_{j=1}^{v_q} X_j \overset{d}{=} X, \quad q \in (0,1), \tag{56}$$

where the X_i's are i.i.d. copies of X. Equivalently, we have

$$X \overset{d}{=} \sum_{j=1}^{v_q} Y_j, \quad q \in (0,1), \tag{57}$$

where $Y_i = qX_i$ has the exponential distribution with parameter λ/q. The latter equality in distribution shows that the exponential distribution is *geometrically infinitely divisible,* (Klebanov et al., 1984). Motivated by the above property of the exponential distribution, we introduce a notion of geometric infinite divisibility for angular distributions.

Definition 2: *An angular r.v. Θ is said to be geometric infinitely divisible if for any* $q \in (0,1)$ *there exist i.i.d. angular r.v. 's* $\Theta_1, \Theta_2, \ldots\ldots\ldots$ *such that*

$$\Theta_1 + \cdots + \Theta_{v_q} (\mathrm{mod}\, 2\pi) \underset{=}{d}\, \Theta, \tag{58}$$

where v_q *has the geometric distribution (55).*

Clearly, geometric infinite divisibility of a real r.v. X implies the same property for the wrapped r.v. X_w. Thus, all wrapped exponential r.v.'s are geometric infinitely divisible.

Proposition 2: if $\Theta \sim WE(\lambda)$, where $\lambda \in R$, then Θ is geometric infinitely divisible. Moreover, for any $q \in (0,1)$ the equality in distribution (58) holds where the Θ_i's have the uniform circular distribution for $\lambda = 0$ and the $WE(\lambda/q)$ distribution for $\lambda \neq 0$.

Proof: First, assume that $\lambda = 0$, in which case Θ has the circular uniform distribution with the ch.f.

$$\phi_p = \frac{e^{2\pi p i} - 1}{2\pi p i}, \quad p \neq 0. \tag{59}$$

Let $\Theta_1, \Theta_2, \ldots\ldots$ be i.i.d. copies of Θ. Conditioning on the distribution of v_q, we can write the ch.f. of the left hand side in (58) as follows :

$$\begin{aligned} E e^{ip(\Theta_1+\ldots+\Theta_{v_q})(\mathrm{mod}\,2\pi)} &= \sum_{k=1}^{\infty} E e^{ip(\Theta_1+\ldots+\Theta_k)(\mathrm{mod}\,2\pi)} q(1-q)^{k-1} \\ &= \sum_{k=1}^{\infty} \phi_p q(1-q)^{k-1} = \phi_p . \end{aligned} \tag{60}$$

where we have used the stability property (52) or the circular uniform distribution. Since we obtained the ch.f. of the circular uniform distribution, we conclude that the representation (58) holds for this case. The proof for the case $\lambda \neq 0$ is similar and we omit it.

Remark: We can restate the conclusion of proposition 2 as follows: if $\Theta_1, \Theta_2, \ldots\ldots$ are i.i.d. with the $WE(\lambda)$ distribution, then the equality in distribution (58) holds, where the r.v. Θ has the $WE(\lambda q)$ distribution. We see that as q converges to zero, then the distribution of $\Theta_1+\ldots\ldots+\Theta_{v_q}$ (mod 2π) converges to the $WE(0)$ distribution, which is the circular uniform distribution. Recall that under mild conditions, the convergence to the circular uniform distribution holds in general if there are n terms in the summation and $n \to \infty$ (Mardia and Jupp, 2000).

Maximum entropy property: The entropy of a r.v. Θ with p.d.f. f is defined as follows :

$$H(\Theta) = -\int_0^{2\pi} f(\theta) \ln f(\theta)\, d\theta , \tag{61}$$

and measures the uncertainty associated with the probability distribution of Θ. A general inference procedure consisting of finding a distribution that maximizes the entropy (61) was proposed by Jaynes (1957), and the method has found the applications in a variety of fields including statistical mechanics stock market analysis, queuing theory, and reliability estimation (Kapur, 1993). It is well known that under no restrictions on f the entropy (61) is maximal for the circular uniform distribution, while if the mean direction and circular variance are fixed, then the von Mises distribution yields the maximal entropy (Kapur, 1993). It turns out that the wrapped exponential distribution $WE(\lambda)$ maximizes the entropy under the constrain that the mean be fixed,

$$\int_0^{2\pi} \theta f(\theta) d\theta = m, \quad 0 < m < 2\pi \tag{62}$$

Proposition 3: *Consider the class C of all the circular r.v 's with density f satisfying the condition (62). Then, the maximum entropy is attained by the WE (λ) distribution with density (8), where $\lambda=(2\pi\xi)^{-1}$ and ξ satisfies the equation*

$$\frac{m}{2\pi} = \xi - \frac{1}{e^{1/\xi} - 1} \tag{63}$$

Moreover the maximal entropy is

$$\max_{\Theta \in c} H(\Theta) = \ln\left(\frac{1-e^{-2\pi\lambda}}{\lambda}\right) + \frac{1}{\lambda} - \frac{2\pi e^{-2\pi\lambda}}{1-e^{-2\pi\lambda}} \tag{64}$$

Proof: The result follows from the fact that under the condition (62) the entropy is maximized by truncated exponential distribution with density $Ce^{-\lambda\theta}$ ($\theta \in [0, 2\lambda)$) with the value of λ specified above (Kapur, 1993), since as we observed earlier truncated exponential density coincides with the wrapped exponential density on the interval $[0, 2\pi)$. The value of the maximum entropy (64) is obtained by straightforward integration in (61) where f is the wrapped exponential density (8).

Remark: Since the function $g(\xi) = \xi - (e^{1/\xi} - 1)^{-1}$ is monotonically increasing from 1/2 to 1 on the interval $(-\infty, 0)$, and monotonically increasing from 0 to 1/2 on the inverval $(0, \infty)$, the distribution maximizing the entropy (61) under the restriction (62) is wrapped exponential $(\lambda > 0)$ for $0 < m < \pi$, wrapped negative exponential $(\lambda < 0)$ for $0 < m < 2\pi$, and circular uniform $(\lambda = 0)$ for $m = \pi$. θ∈

Remark: The above result is analogous to the well-known maximum entropy characterization of the exponential distribution (which maximizes the entropy among all continuous probability distribution on $(0, \infty)$, with a given mean (Kapur, 1993)).

Estimation

Here we consider the problem of estimating the parameter λ of a general wrapped exponential distribution. Let $\theta_1, ..., \theta_n$ be random sample from *WE* (λ) distribution with p.d.f.

$$f_\lambda(\theta) = \frac{\lambda e^{-\lambda\theta}}{1-e^{-2\pi\lambda}}, \quad \theta \in [0, 2\pi), \lambda \in R \tag{65}$$

In casel $\lambda = 0$ the distribution is understood in the limiting case where $f_0(\theta) = 1/2\pi$, $\theta \in [0, 2\pi)$ (circular uniform distribution).

The maximum likelihood estimation: Let us define the statistic

$$\tilde{\theta} = \frac{1}{2\pi\, n}\sum_{j=1}^{n} \theta_j , \tag{66}$$

where $\bar{\theta} \in (0,1)$. Then, the likelihood function takes the form

$$L(\lambda) = \prod_{j=1}^{n} f_{\lambda}(\theta_j) = \frac{\lambda^{n} e^{-2\pi n \lambda \bar{\theta}}}{(1 - e^{-2\pi\lambda})^{n}}, \quad (67)$$

while the log-likelihood is

$$g(\lambda) = \ln L(\lambda) = n \ln \lambda - n \ln\left(1 - e^{-2\pi\lambda}\right) - 2\pi n \lambda \bar{\theta}. \quad (68)$$

Our objective is to find the value of λ that maximizes the log-likelihood function g. The first derivative of g is

$$\frac{d}{d\lambda} g(\lambda) = 2\pi n\left(h(2\pi\lambda) - \bar{\theta}\right). \quad (69)$$

where the real function h is defined as follows

$$h(x) = 1/x - 1/(e^{x} - 1). \quad (70)$$

The following simple result summarizes those properties of h that are relevant to our problem.

Lemma 2: *The function h defined by (70) is continuous and strictly decreasing on $(-\infty, \infty)$ and onto $(0,1)$ with $h(0) = 1/2$, $\lim_{x\to\infty} h(x) = 0$, and $\lim_{x\to-\infty} h(x) = 1$.*

Proof: First, note that clearly $\lim_{x\to\infty} h(x) = 0$, and $\lim_{x\to-\infty} h(x) = 1$. The value at zero, $h(0) = 1/2$, follows by D' Hospital's rule applied to the limit $\lim_{x\to 0} h(x)$. It remains to show that the function h is strictly decreasing. It is enough to show that $h'(x) < 0$ for $x \neq 0$. The latter inequality becomes

$$h'(x) = \frac{e^{x}}{\left(e^{x} - 1\right)^{2}} - \frac{1}{x^{2}} < 0. \quad (71)$$

We assume that $x > 0$, since the case $x < 0$ is quite similar. Denoting $y = e^{x} - 1 > 0$, we have

$$\ln^{2}(y+1) < \frac{y^{2}}{y+1}, \quad y > 0, \quad (72)$$

or, taking the roots,

$$\ln(y+1) < \frac{y}{\sqrt{y+1}}, \quad y > 0. \quad (73)$$

Since the two functions $h_1(y) = \ln(y+1)$ and $h_2(y) = y/\sqrt{y+1}$ are continuous and differentiable with $h_1(0) = h_2(0) = 0$, the inequality (72) will be established if we can show that

$$\frac{d}{dy} h_1(y) < \frac{d}{dy} h_2(y), \qquad y > 0. \tag{74}$$

Upon taking the derivatives, (74) becomes

$$\frac{1}{1+y} < \frac{1+y/2}{(1+y)\sqrt{1+y}}, \qquad y > 0, \tag{75}$$

which is equivalent to

$$\sqrt{1+y} < 1 + y/2, \qquad y > 0. \tag{76}$$

After squaring each side, the latter inequality yields

$$1 + y < 1 + y + y^2/4, \qquad y > 0, \tag{77}$$

which holds trivially. Finally, in the case $x < 0$, denote $y = 1 - e^x > 0$ and proceed as in the previous case.

The wrapped exponential case: Assume that the sample is from the wrapped exponential model $WE(\lambda)$, $\lambda \geq 0$. If the statistic $\tilde{\theta} > 1/2$, then by (69) and Lemma 2, the first derivative of the log-likelihood function is negative for all $\lambda \geq 0$. Thus, the likelihood is strictly decreasing on $[0, \infty)$ and the maximum value is attained at $\lambda = 0$. On the other hand, for $0 \leq \tilde{\theta} \leq 1/2$, there will be a unique point $\hat{\lambda} \in [0, \infty)$ such that the log-likelihood function g increases on $[0, \hat{\lambda})$ and decreases on $(\hat{\lambda}, \infty)$, and consequently, such point will be the maximum likelihood estimate of λ. The following result summarizes this discussion.

Proposition 4: *Let $\theta_1, \ldots, \theta_n$ be a random sample from the $WE(\lambda)$ distribution with p.d.f. (65), where $\lambda \geq 0$. Then, the maximum likelihood estimate $\hat{\lambda}$ of λ is unique and is given by*

(a) $\hat{\lambda} = 0$ if $\tilde{\theta} > 1/2$;

(b) $\hat{\lambda} = 1/2\pi\, h^{-1}(\tilde{\theta})$ if $0 \leq \tilde{\theta} \leq 1/2$;

where the statistic $\tilde{\theta}$ is given by (66) and h^{-1} is the inverse of h given by (70).

The wrapped negative exponential case: If the sample is from the $WE(\lambda)$ distribution with $\lambda \leq 0$ (wrapped negative exponential model), then for $\tilde{\theta} < 1/2$, the first derivative of the log-likelihood function is positive for all $\lambda \le 0$, and consequently the MLE of λ is equal to zero. For $\tilde{\theta} \ge 1/2$, the MLE of λ is the same as before.

Proposition 5: *Let $\theta_1, ..., \theta_n$ be a random sample from the WE (λ) distribution with p.d.f. (65), where $\lambda \le 0$. Then, the maximum likelihood estimate $\hat{\lambda}$ of λ is unique and is given by*

$$\text{(a)}\quad \hat{\lambda} = 0 \text{ if } \tilde{\theta} < 1/2;$$

$$\text{(b)}\quad \hat{\lambda} = 1/2\pi\, h^{-1}\left(\tilde{\theta}\right) \text{ if } ½ \le \tilde{\theta}\ 1½;$$

where the statistic $\tilde{\theta}$ is given by (66) and h^{-1} is the inverse of h given by (70).

The general case : If the sample is from the general *WE* (λ) distribution with $\lambda \in R$, then the MLE of λ is equal to zero only if $\tilde{\theta} = 1/2$.

Proposition 6: Let $\theta_1, ..., \theta_n$ be a random sample from the *WE* (λ) distribution with p.d.f (65), where $\lambda \in R$. Then, the unique maximum likelihood estimate of λ is $\hat{\lambda} = \frac{1}{2\pi} h^{-1}(\tilde{\theta})$, where $\tilde{\theta}$ and h^{-1} are as before.

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